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Fluctuations and Phase Transition Dynamics

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Kibble and Zurek have provided a unifying causal picture for the appearance of classical defects like cosmic strings or vortices at the onset of phase transitions in relativistic QFT and condensed matter systems, respectively. In condensed matter the predictions are partially supported by agreement with experiments in superfluid helium. We provide an alternative picture for the initial appearance of defects that supports the experimental evidence. When the original predictions fail, this is understood, in part, as a consequence of thermal fluctuations (noise), which play a comparable role in both condensed matter and QFT.

1. OVERVIEW

In this paper I consider the emergence of "classical" field configurations—topological defects—after a phase transition, and the extent to which thermal fluctuations can inhibit this process. This is of particular interest in the early universe, for which we expect a sequence of transitions from a very symmetric initial state, and in which the presence of classical defects can have important astrophysical consequences.

The relevance of topological defects is that when symmetry breaks, it does not do so uniformly. At the very least, the field is uncorrelated on the scale of the causal horizon at any time. Since a broken symmetry is, necessarily, characterized by degenerate vacua, the choice of different vacua in domains in which the fields are uncorrelated will lead naturally to topological defects between them as the field does its best to order on large scales. The nature of the defects depends on the relevant homotopy group of the ground-state manifold. The most acceptable defect on cosmological grounds is the "cosmic string"—a generalized field vortex—which may have played in role in structure formation.

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The details do not concern us here. What interests us is how this collection of essentially classical objects, which can be observed directly in principle, came into existence. The simplest question, which we shall address here, is what is the density of cosmic strings (or other defects) at the time of their appearance?

The early universe is very hot, but such a problem requires us to go beyond equilibrium thermal field theory. In practice, we often know remarkably little about the dynamics of thermal systems. For simplicity, I shall assume scalar field order parameters, with *continuous* transitions. In principle, the field correlation length diverges at a continuous transition. In practice, it does not since there is not enough time. One possibility is that the separation of "defects" is characterized by the correlation length when it checks its growth. If this were simply so, a measurement of defect densities would be a measurement of correlation lengths. Estimates of this early field ordering and its contingent defects in the early universe have been made by Kibble [1, 2], using simple thermal [1] arguments or causal arguments [2] different from the one above (although that is also due to Kibble [3]).

There are great difficulties in converting such predictions for the early universe into experimental observations since, but for a possible stray monopole, we have no direct evidence for them having existed.[2] Further, if vortices evolve into networks that show scaling, then length distributions are of greater importance than density. However, Zurek suggested [4] that similar arguments to those in ref. 2 were applicable to condensed matter systems for which direct experiments on defect densities could be performed. This has led to considerable activity from theorists working on the boundary between QFT and condensed matter theory and from condensed matter experimentalists. To date almost all experiments have involved superfluids, for which vortices can be produced readily. All but one experiment is in agreement with these simple causal predictions and we shall pay particular attention to this one failure of prediction. In this paper I do the following:

- Review the Kibble/Zurek causality predictions for initial correlation lengths and defect densities.
- summarize the results of the condensed matter experiments and present an alternative picture in which thermal noise is explicit. I shall then show how this alternative picture gives essentially the same results as the Zurek picture for those condensed matter systems for which there is experimental agreement.
- Use these ideas to address the more complicated problem of the

[2] Although this does not impede our ability to make predictions for defect-driven fluctuations in the CMB, for example.

appearance of "classical" defect configurations in QFT in the light of Kibble's predictions, and the role of thermal noise in them.

A more detailed discussion of these and other issues is given elswhere [5].

2. WHEN SYMMETRY BREAKS, HOW BIG ARE THE SMALLEST IDENTIFIABLE PIECES?

Defects in the large-scale ordering of the field can only appear once the transition has taken place. If it is the case that defect density can be identified simply from the field correlation length, the *maximum* density (an experimental observable in condensed matter systems, although not for the early universe) will be associated with the *smallest* identifiable correlation length in the broken phase once the transition has been effected.

In order to see how to identify these "smallest pieces"[3] it is sufficient to consider the simplest theory with vortices, that of a single relativistic *complex* scalar field in three spatial dimensions, undergoing a temperature quench. In the first instance we assume that the qualitative dynamics of the transition are conditioned by the field's *equilibrium* free energy of the form

$$F(T) = \int d^3x \, (|\nabla\phi|^2 + m^2(T)|\phi|^2 + \lambda|\phi|^4) \tag{1}$$

Prior to the transition, at temperature $T > T_c$, the critical temperature, $m(T) > 0$ plays the role of an effective "plasma" mass due to the interactions of ϕ with the heat bath, which includes its own particles. After the transition, when T is effectively zero, $m^2(0) = -M^2 < 0$ enforces the $U(1)$ symmetry breaking, with field expectation values $\langle|\phi|\rangle = \eta$, $\eta^2 = M^2/\lambda$. The change in temperature that leads to the change in the sign of m^2 is most simply understood as a consequence of the universe expanding. Models that attempt to take inflation into account, however, lead to "preheating" that is not Boltzmannian [7]. Nonetheless, even in such cases it is possible to isolate an effective temperature for long-wavelength modes. This is all that is necessary, but is too sophisticated for the simple scenarios that we shall present here. We shall not even include a metric in Eq. (1).

The minima of the final potential of Eq. (1) now constitute the circle $\phi = \eta e^{i\alpha}$. When the transition starts, ϕ begins to fall into the valley of the potential, choosing a random phase subject to continuity. At late times the failure of the field to be uniform in phase on large scales will lead to it twisting around classical "defects"—solutions to $\delta F/\delta\phi = 0$ that locally minimize the energy stored in field gradients and potentials. Those of interest

[3] The title of this section is essentially that posed in recent papers by Zurek [6].

to us are vortices, tubes of "false" vacuum $\phi \approx 0$, around which the field phase changes by $\pm 2\pi$. In an early universe context these are the simplest possible "cosmic strings."

How this collapse takes place determines the size of the first identifiable domains. It was suggested by Kibble and Zurek that this size is essentially the equilibrium field correlation length $\xi_{\rm eq}$ at some appropriate temperature close to the transition. I shall argue later that this is too simple, but it is a plausible starting point. Two very different mechanisms have been proposed for estimating this size.

2.1. Thermal Activation

In the early work on the cosmic string scenario an alternative possibility to simple causality was to assume [1] that initial domain size was fixed in the Ginzburg regime by the correlation length at that time, rather than the causal radius. By this we mean the following. Once we are below T_c and the central hump in $V(\phi) = m^2(T)|\phi|^2 + \lambda|\phi|^4$ is forming, T_G signals the temperature above which there is a significant probability for thermal fluctuations over the central hump on the scale of the correlation length. Most simply, it is determined by the condition

$$\Delta V(T_G)\xi_{\rm eq}^3(T_G) \approx T_G \tag{2}$$

where $\Delta V(T)$ is the difference between the central maximum and the minima of $V(\phi, T)$. We find $|1 - T_G/T_c| = O(\lambda)$.

Below T_G fluctuations from one minimum to the other become increasingly unlikely. When this happens the correlation length is

$$\xi_{\rm eq}(T_G) = O\left(\frac{\xi_0}{\sqrt{1 - T_G/T_c}}\right) \tag{3}$$

where $\xi_0 = M^{-1}$ is the natural unit of length, the Compton wavelength of the ϕ particles.

It is tempting [1, 8] to identify $\xi_{\rm eq}(T_G)$ with the scale at which stable domains begin to form. We shall see later that this is incorrect for quenches that are not too slow. However, some care is needed if (as can happen in condensed matter physics) we never leave the Ginzburg regime.

The formation of large domains is an issue that requires more than equilibrium physics. The simplest dynamical arguments can be understood in terms of causality.

2.2. Causality

We have already mentioned that causality puts an *upper* bound on domain size. Specifically, if $G(r, t)$ is the two-field correlation function at time t for separation r, then G vanishes for $r \geq 2t$ approximately. This was used by Kibble [3] to put an upper bound on monopole density in the early universe. If this causal bound and the Ginzburg criteria attempt to set scales once the critical temperature has been *passed*, the causal arguments considered now attempt to set scales *before* it is reached.

Here we attempt a *lower* bound on domain size, an upper bound on defect density. Suppose the temperature $T(t)$ varies sufficiently slowly with time t that it makes sense to replace $V(\phi, T)$ by $V(\phi, T(t))$. With $m^2(T(t))$ vanishing at $T = T_c$, which we suppose happens at $t = 0$, the *equilibrium* correlation length of the field fluctuations $\xi_{\text{eq}}(T(t)) = |m^{-1}(T(t))|$ diverges at $T(t) = T_c$. It is sufficient to adopt a mean-field approximation in which $m^2(T) \propto (T - T_c)$. The true correlation length $\xi(t)$ cannot diverge like $\xi_{\text{eq}}(T(t))$, since it can only grow so far in a finite time.

Initially, for $t < 0$, when we are far from the transition, we again assume effective equilibrium, and the field correlation length $\xi(t)$ tracks $\xi_{\text{eq}}(T(t))$ approximately. However, as we get closer to the transition, $\xi_{\text{eq}}(T(t))$ begins to increase arbitrarily fast. As a crude upper bound, the true correlation length fails to keep up with $\xi_{\text{eq}}(T(t))$ by the time $-\bar{t}$ at which ξ_{eq} is growing at the speed of light, $d\xi_{\text{eq}}(T(-\bar{t}))/dt = 1$. It was suggested by Kibble [2] that, once we have reached this time, $\xi(t)$ *freezes* in, remaining approximately constant until the time $t \approx +\bar{t}$ after the transition, when it once again becomes comparable to the now decreasing value of ξ_{eq}. The correlation length $\xi_{\text{eq}}(\bar{t}) = \xi_{\text{eq}}(-\bar{t})$ is argued to provide the scale for the minimum domain size *after* the transition.

Specifically, if we assume a time dependence $m^2(t) = -M^2 t/t_Q$ in the vicinity of $t = 0$, when the transition begins to be effected, then the causality condition gives $t_C = t_Q^{1/3}(2M)^{-2/3}$. As a result, $M\xi_{\text{eq}}(\bar{t}) = (M\tau_0)^{1/3}$, which, with condensed matter in mind, we write as

$$\bar{\xi} = \xi_{\text{eq}}(\bar{t}) = \xi_0\left(\frac{\tau_Q}{\tau_0}\right)^{1/3} \tag{4}$$

where $\tau_0 = \xi_0 = M^{-1}$ are the natural time and distance scales. In contrast to Eq. (3), Eq. (4) depends explicitly on the quench rate, as we would expect.

2.3. QFT or Condensed Matter

This approach of Kibble was one of the motivations for a similar analysis by Zurek [4] of transitions in condensed matter. After rescaling, F could equally well be the Ginzburg–Landau free energy for the complex order-parameter field whose magnitude determines the superfluid density. That is,

$$F(T) = \int d^3x \left(\frac{\hbar^2}{2m} |\nabla\phi|^2 + \alpha(T)|\phi|^2 + \frac{1}{4}\beta|\phi|^4 \right) \tag{5}$$

in which $\alpha(T) \propto m^2(T)$ vanishes at the critical temperature T_c. The only difference is that, in the causal argument, the speed of light should be replaced by the speed of (second) sound, with different critical index.

Explicitly, let us assume the mean-field result $\alpha(T) = \alpha_0\epsilon(T)$, where $\epsilon = (T/T_c - 1)$, remains valid as T/T_c varies with time t. In particular, we first take $\alpha(t) = \alpha(T(t)) = -\alpha_0 t/\tau_Q$ in the vicinity of T_c. The fundamental length scale ξ_0 is given from Eq. (5) as $\xi_0^2 = \hbar^2/2m\alpha_0$. The Gross–Pitaevski theory suggests a natural time scale $\tau_0 = \hbar/\alpha_0$. When we later adopt the time-dependent Landau–Ginzburg (TDLG) theory we find this still to be true, empirically, at order-of-magnitude level, and we keep it.

It follows that the equilibrium correlation length $\xi_{\text{eq}}(t)$ and the relaxation time $\tau(t)$ diverge when t vanishes as

$$\xi_{\text{eq}}(t) = \xi_0 \left|\frac{t}{\tau_Q}\right|^{-1/2}, \qquad \tau(t) = \tau_0 \left|\frac{t}{\tau_Q}\right|^{-1} \tag{6}$$

The speed of sound is $c(t) = \xi_{\text{eq}}(t)/\tau(t)$, slowing down as we approach the transition as $|t|^{1/2}$. The causal counterpart to $d\xi_{\text{eq}}(t)/dt = 1$ for the relativistic field is $d\xi_{\text{eq}}(t)/dt = c(t)$. This is satisfied at $t = -\bar{t}$, where $\bar{t} = \sqrt{\tau_Q\tau_0}$, with corresponding correlation length

$$\bar{\xi} = \xi_{\text{eq}}(\bar{t}) = \xi_{\text{eq}}(-\bar{t}) = \xi_0 \left(\frac{\tau_Q}{\tau_0}\right)^{1/4} \tag{7}$$

[cf. Eq. (4)]. We stress that, yet again, the assumption is that the length scale that determines the initial correlation length of the field freezes in *before* the transition begins.

3. EXPERIMENTS

The jump that Kibble made [2] in QFT was to assume that the correlation length, Eq. (4), also sets the scale for the typical minimum intervortex distance. That is, the *initial* vortex density n_{def} is[4] is assumed to be

$$n_{\text{def}} = \frac{1}{f^2}\frac{1}{\bar{\xi}^2} = \frac{1}{f^2\xi_0^2}\left(\frac{\tau_0}{\tau_Q}\right)^{2\gamma} \tag{8}$$

for $\gamma = 1/3$ and $f = O(1)$. We stress that this assumption is *independent* of the argument that led to Eq. (4). Since ξ_0 also measures cold vortex thickness,

[4] Equivalently, the length of vortices in a box of volume v is $O(n_{\text{def}}v)$.

$\tau_Q \gg \tau_0$ corresponds to a measurably large number of widely separated vortices.

Even if cosmic strings were produced in so simple a way in the very early universe it is not possible to compare the density, Eq. (8), with experiment, in large part because of our uncertainty as to what is the appropriate theory. It was Zurek who first suggested that this causal argument for defect density be tested in condensed matter systems.

3.1. Superfluid Helium

Vortex lines in both superfluid ^{4}He and ^{3}He are good analogues of global cosmic strings. In ^{4}He the Bose superfluid is characterized by a complex field ϕ, whose squared modulus $|\phi|^2$ is the superfluid density. The Landau–Ginzburg theory for ^{4}He has, as its free energy, $F(T)$ of Eq. (5). The static classical field equation $\delta F/\delta\phi = 0$ has vortex solutions as before with effective width ξ_0.

The situation is more complicated, but more interesting, for *fermionic* ^{3}He. Somewhat as in a BCS superconductor, these fermions form the counterpart to Cooper pairs. However, whereas the (electron) Cooper pairs in a superconductor form a ^1S state, the ^{3}He pairs form a ^3P state. The order parameter $A_{\alpha i}$ is a complex 3×3 matrix $A_{\alpha i}$. There are two distinct superfluid phases, depending on how the $SO(3) \times SO(3) \times U(1)$ symmetry is broken. If the normal fluid is cooled at low pressures, it makes a transition to the ^{3}He-B phase.

The Landau–Ginzburg free energy is, necessarily, more complicated, permitting many types of vortex [9], but the effective potential $V(A_{\alpha i}, T)$ has the diagonal form [10] $V(A, T) = \alpha(T)|A_{\alpha i}|^2 + O(A^4)$ for small fluctuations, and this is all that we need for the production of vortices at very early times. Thus the Zurek analysis leads to the prediction Eq. (8), as before, for appropriate γ.[5]

3.2. Counting Vortices

Although ^{3}He is more complicated to work with, the experiments to check Eq. (8) are cleaner, since even individual vortices can be detected by magnetic resonance. Second, because the vortex width is many atomic spacings, the Landau–Ginzburg theory is good ($\gamma = 1/4$).

So far, experiments have been of two types. In the Helsinki experiment [11] superfluid ^{3}He in a rotating cryostat is bombarded by slow neutrons. Each neutron entering the chamber releases 760 keV via the reaction $n + {}^3\text{He} \to p + {}^3\text{He} + 760$ keV. The energy goes into the kinetic energy of the

[5] For ^{4}He, mean-field theory is poor, and a better value for γ is $\gamma = 1/3$.

proton and triton, and is dissipated by ionization, heating a region of the sample above its transition temperature. The heated region then cools back through the transition temperature, creating vortices. Vortices above a critical size grow and migrate to the center of the apparatus, where they are counted by an NMR absorption measurement. The quench is very fast, with $\tau_Q/\tau_0 = O(10^3)$. Agreement with Eqs. (7) and (8) is good. This is even though it is now argued [12] that the Helsinki experiment should *not* show agreement because of the geometry of the heating event.

The second type of experiment has been performed at Grenoble and Lancaster [13]. Rather than count individual vortices, the experiment detects the total energy going into vortex formation when ^{3}He is irradiated by neutrons. After each absorption the energy released in the form of quasiparticles is measured, and found to be less than the total 760 keV. This missing energy is assumed to have been expended on vortex production. Again, agreement with Zurek's prediction, Eqs. (7) and (8), is good.

The experiments in ^{4}He conducted at Lancaster follow Zurek's original suggestion. The idea is to expand a sample of normal fluid helium so that it becomes superfluid at essentially constant temperature. That is, we change $1 - T/T_c$ from negative to positive by reducing the pressure and increasing T_c. As the system goes into the superfluid phase a tangle of vortices is formed because of the random distribution of field phases. The vortices are detected by scattering second sound off them. A mechanical quench is slow, with τ_Q some tens of milliseconds, and $\tau_Q/\tau_0 = O(10^{10})$. Two experiments have been performed [14, 15]. In the first fair agreement was found with the prediction of Eq. (8), but the second experiment failed to see any vortices whatsoever.

There is certainly no agreement, in this or any other experiment on ^{3}He, with the thermal fluctuation density that would be based on Eq. (3).

4. THE KIBBLE–ZUREK PICTURE FOR THE VALUE OF $\bar{\xi}$ IS CORRECT

To do better than the simple causality arguments we need a concrete model for the dynamics.

4.1. Condensed Matter: The TDLG Equation at Early Times

We assume that the dynamics of the transition can be derived from the explicitly time-dependent Landau–Ginzburg free energy

$$F(t) = \int d^3x \left(\frac{\hbar^2}{2m} |\nabla\phi|^2 + \alpha(t)|\phi|^2 + \frac{1}{4}\beta|\phi|^4 \right) \tag{9}$$

obtained from Eq. (5) on identifying $\alpha(t) = \alpha(T(t)) = \alpha_0\epsilon(t)$, where $\epsilon =$

$(T/T_c - 1)$. In a quench in which T_c or T changes it is convenient to shift the origin in time, to write $\epsilon(t)$ as

$$\epsilon(t) = \epsilon_0 - \frac{t}{\tau_Q}\,\theta(t) \tag{10}$$

for $-\infty < t < \tau_Q(1 + \epsilon_0)$, after which $\epsilon(t) = -1$. Here ϵ_0 measures the original relative temperature and τ_Q defines the quench rate. The quench begins at time $t = 0$ and the transition from the normal to the superfluid phase begins at time $t_0 = \epsilon_0\tau_Q$.

Motivated by Zurek's later numerical [6] simulations, we adopt the time-dependent Landau-Ginzburg (TDLG) equation for F, on expressing ϕ as $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, such that

$$\frac{1}{\Gamma}\frac{\partial\phi_a}{\partial t} = -\frac{\delta F}{\delta\phi_a} + \eta_a \tag{11}$$

where η_a is Gaussian thermal noise satisfying

$$\langle\eta_a(\mathbf{x}, t)\eta_b(\mathbf{y}', t')\rangle = 2\delta_{ab}T(t)\Gamma\delta(\mathbf{x} - \mathbf{y})\delta(t - t') \tag{12}$$

This is a crude approximation for ^{4}He, and a simplified form of a realistic description of ^{3}He, but it is not a useful description of QFT, as it stands.

It is relatively simple to determine the validity of Zurek's argument since it assumes that freezing-in of field fluctuations occurs just before symmetry breaking begins. At that time the self-interaction term can be neglected ($\beta = 0$). In space, time, and temperature units in which $\xi_0 = \tau_0 = k_B = 1$, Eq. (11) then becomes

$$\dot{\phi}_a(\mathbf{x}, t) = -[-\nabla^2 + \epsilon(t)]\phi_a(\mathbf{x},t) + \bar{\eta}_a(\mathbf{x},t) \tag{13}$$

where $\bar{\eta}$ is the renormalized noise. The solution of the "free"-field linear equation is straightforward, giving a Gaussian equal-time correlation function [16, 17]

$$\langle\phi_a(\mathbf{r}, t)\phi_b(\mathbf{0}, t)\rangle = \delta_{ab}G(\mathbf{r}, t) \tag{14}$$

where

$$G(r, t) = \int_0^\infty d\tau\, \bar{T}(t - \tau/2)(1/(4\pi\tau))^{3/2} \times \exp(-r^2/4\tau)\exp\left[-\int_0^\tau ds\, \epsilon(t - s/2)\right] \tag{15}$$

and $\bar{T}$ is the renormalized temperature. At time $t_0 = \epsilon_0\tau_0$, when the transition begins, a saddle-point calculation shows that, provided the quench is not too fast,

$$G(r, t_0) \approx \frac{T_c}{4\pi r}\, e^{-a(\tau/\bar{\xi})^{4/3}} \tag{16}$$

where $a = O(1)$, confirming Zurek's result, Eq. (7).

4.2. QFT: Closed Time-Path Ensemble Averaging at Early Times

For QFT the situation is rather different. In the previous section, instead of working with the TDLG equation, we could have worked with the equivalent Fokker–Planck equation for the probability $p_t^{\mathrm{FP}}[\Phi]$ that, at time $t > 0$, the measurement of ϕ will give the function $\Phi(\mathbf{x})$. Thus $G(r,t)$ of Eq. (14) can be written as

$$\delta_{ab}G(\mathbf{r}, t) = \langle \phi_a(\mathbf{r}, t)\phi_b(\mathbf{0}, t)\rangle = \int \mathcal{D}\Phi\; p_t^{\mathrm{FP}}[\Phi]\Phi_a(\mathbf{r})\Phi_b(\mathbf{0}) \tag{17}$$

When solving the dynamical equations for a hot quantum field it is convenient to work with probabilities from the start. Taking $t = 0$ as our starting time for the evolution of the complex field ϕ, suppose that at this time the system is in a pure state, in which the measurement of ϕ would give $\Phi_0(\mathbf{x})$. That is,

$$\hat{\phi}(t = 0, x)\big|\Phi_0, t = 0\rangle = \Phi_0\big|\Phi_0, t = 0\rangle \tag{18}$$

The probability $p_t[\Phi]$ that at time $t > 0$ the measurement of ϕ will give the function $\Phi(\mathbf{x})$ is the double path integral

$$p_t[\Phi] = \int_{\phi_\pm(0)=\Phi 0}^{\phi_\pm(t)=\Phi} \mathcal{D}\phi_+\, \mathcal{D}\phi_-\, \exp\{i(S_t[\phi_+] - S_t[\phi_-])\} \tag{19}$$

where $\mathcal{D}\phi_\pm = \mathcal{D}\phi_{\pm,1}\mathcal{D}\phi_{\pm,2}$ and $S_t[\phi]$ is the (time-dependent) action obtained from Eq. (1) on substituting $m(t) = m(T(t))$ for $m(T)$.

$p_t[\Phi]$ can be written in the closed-time-path form in which, instead of separately integrating $\phi_\pm$ along the time paths $0 \leq t \leq t_f$, the integral can be interpreted as the time-ordering of a field ϕ along the closed path $C_+ \oplus C_-$ of Fig. 1, where $\phi = \phi_+$ on C_+ and $\phi = \phi_-$ on C_-. When we extend the contour from t_f to $t = \infty$ either ϕ_+ or ϕ_- is an equally good candidate for the physical field, but we choose ϕ_+.

Rather than assume a pure state at time $t = 0$, we assume that Φ is Boltzmann distributed at time $t = 0$ at an effective temperature of $T_0 = \beta_0^{-1}$ according to the Hamiltonian $H[\Phi]$ corresponding to the free-field action $S_0[\phi]$, obtained by setting $\lambda = 0$ in Eq. (1), in which ϕ is taken to be periodic in imaginary time with period β_0. We now have the explicit form for $p_t[\Phi]$,

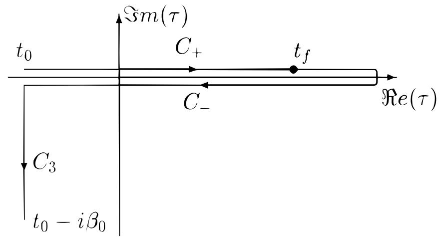

Fig. 1. The closed-time-path contour $C_+ \oplus C_-$, with the Boltzmann imaginary leg.

$$p_t[\Phi] = \int_B \mathcal{D}\phi \; e^{iS_C[\phi]}\delta[\phi_+(t_f) - \Phi] \tag{20}$$

written as the time ordering of a single field along the contour $C = C_+ \oplus C_- \oplus C_3$, extended to include a third imaginary leg, where ϕ takes the values ϕ_+, ϕ_-, and ϕ_3 on C_+, C_-, and C_3, respectively, for which S_C is $S[\phi_+]$, $S[\phi_-]$, and $S_0[\phi_3]$.

Just as we had no need to calculate $p_t^{FP}[\Phi]$ explicitly in condensed matter, we can average in QFT without having to calculate $p_t[\Phi]$ explicitly. Specifically,

$$G_{ab}(r, t) = \langle\Phi_a(\mathbf{r})\Phi_b(\mathbf{0})\rangle_t = \int \mathcal{D}\Phi \; p_t[\Phi]\Phi_a(\mathbf{r}) \; \Phi_b(\mathbf{0}) \tag{21}$$

is given by

$$G_{ab}(r, t) = \langle\phi_a(\mathbf{r}, t)\phi_b(\mathbf{0}, t)\rangle \tag{22}$$

which is the equal-time thermal Wightman function with the given thermal boundary conditions.

Fortunately, as for the condensed matter case, the interval $-\bar{t} \leq t - t_0 \leq \bar{t}$ occurs in the *linear* regime, when the self-interactions are unimportant. The relevant equation for constructing the correlation functions of this one-field system is now the second-order equation

$$\frac{\partial^2\phi_a}{\partial t^2} = -\frac{\delta F}{\delta\phi_a} \tag{23}$$

for F of Eq. (1). This is solvable in terms of the mode functions $\chi_k^{\pm}(t)$, identical for $a = 1, 2$, satisfying

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 + m^2(t)\right]\chi_k^{\pm}(t) = 0 \tag{24}$$

subject to $\chi_k^{\pm}(t) = e^{\pm i\omega_{\text{in},t}}$ at $t \leq 0$, for incident frequency $\omega_{\text{in}} = \sqrt{\mathbf{k}^2 + \epsilon_0 M^2}$ and $m^2(t) = \epsilon(t)M^2$, where $\epsilon(t)$ is parametrized as for the TDLG equation above. This corresponds to a temperature quench from an initial state of thermal equilibrium at temperature $T_0 > T_c$, where $(T_0/T_c - 1) = \epsilon_0$. The diagonal correlation function $G(r, t)$ of Eq. (14) is given as the equal-time propagator

$$G(r, t) = \int d\!\!{}^-{}^3k \; e^{i\mathbf{k.x}} \chi_k^+(t) \chi_k^-(t) C(k) \tag{25}$$

where $C(k) = \coth[\omega_{\rm in}(k)/2T_0]/2\ \omega_{\rm in}(k)$ encodes the initial conditions.

An exact solution can be given [18] in terms of Airy functions. Dimensional analysis shows that, on ignoring the k dependence of $C(k)$, appropriate for large r (or small k), $\xi_{\rm eq}(\bar{t})$ of Eq. (4) again sets the scale of the equal-time correlation function. Specifically,

$$G(r, t_0) \propto \int d\kappa \, \frac{\sin \kappa(r/\bar{\xi})}{\kappa(r/\bar{\xi})} \, F(\kappa) \tag{26}$$

where $F(0) = 1$ and $F(\kappa) \sim \kappa^{-3}$ for large κ. Kibble's insight is correct, at least for this case of a single (uncoupled) field.

5. VORTEX DENSITIES DO NOT DETERMINE CORRELATION LENGTHS DIRECTLY

We have seen that there is no reason to disbelieve the causal arguments of Kibble for QFT and Zurek for condensed matter as to the field correlation length at the time of the transition. The excellent agreement with the ^{3}He experiments also shows that, despite the very interesting simulations of Aranson *et al.* [12], this length does, indeed, characterize vortex separation for condensed matter at the time when the defects form.

However, the recent Lancaster experiment shows that this cannot always be the case. Significantly, for ^{3}He the Ginzburg regime is extremely narrow, whereas for ^{4}He it is very broad. In fact, the ^{4}He experiments begin and end in the Ginzburg regime, where thermal fluctuations dominate. The causality arguments are too simple to accommodate these facts.

If these differences are to be visible in the formalism, it can only be through the way in which we relate vortex density to correlation length. We have already observed that the TDLG equation can be recast as the Fokker–Planck equation, whereby the ensemble averages can be understood as averaging with respect to the probability $p_t[\Phi(\mathbf{x})]$ that, at time t, the field takes value $\Phi(\mathbf{x})$. We can use these probabilities, implicit in the correlation functions, to estimate defect densities.

5.1. Classical Defects in Condensed Matter

It would be foolish to estimate the probability of finding vortex profiles directly. One way is to work through *line zeros*, since vortices have line zeros of the complex field ϕ at the center of their cores. The converse is not true since zeros occur on all scales. However, a starting point for counting vortices in superfluids is to count line zeros of an appropriately coarse-grained field [19]. Not wishing to entertain vortices within vortices, we put a cutoff $l = O(\xi_0)$ by hand into the Fourier transform $G(k, t)$ of $G(r, t)$, as

$$G_l(r, t) = \int đ^3k \, e^{i\mathbf{k}.\mathbf{r}} \, G(k, t) \, e^{-k^2 l^2} \tag{27}$$

We stress that the *long-distance* correlation length $\xi_{eq}(\bar{t})$ depends essentially on the position of the nearest singularity of $G(k, t)$ in the complex k-plane, *independent* of l.

This is not the case for the line-zero density n_{zero}. For example, in our Gaussian approximation of the previous section n_{zero} can be calculated exactly from the two-point correlation function $G(r, t)$ with $p_t[\Phi]$ implicit. It can be shown quite easily [20, 21] that it depends on the *short-distance* behavior of $G_l(r, t)$ as

$$n_{zero}(t) = \frac{-1}{2\pi} \frac{G_l''(0, t)}{G_l(0, t)} \tag{28}$$

i.e., the ratio of fourth to second moments of $G(k, t) \, e^{-k^2 l^2}$.

There are several prerequisites before line zeros can be identified with vortex cores, and $n_{zero}(t)$ with $n_{def}(t)$.

1. The field, on average must have achieved its symmetry-broken ground-state equilibrium value

$$\langle |\phi|^2 \rangle = \alpha_0/\beta \tag{29}$$

This in itself is sufficient to show that the causal time $\bar{t}$ is *not* the time to begin looking for defects since $\langle |\phi|^2 \rangle$ is small at this time. This, in turn, requires that $G(k, t)$ be nonperturbatively (in β) large.

2. Only when $\partial n_{zero}/\partial l$ is small in comparison to n_{zero}/l at $l = \xi_0$ will the line zeros have the nonfractal nature of classical defects on small scales, although vortices may behave like random walks on larger scales. As the power in the long-wavelength modes increases, the "Bragg" peak develops in $k^2 G(k, t)$, moving in toward $k = 0$. This condition then becomes the condition that the peak dominates its tail.

3. The energy in field gradients should be commensurate with the energy in classical vortices with same density as that of line zeros.

We stress that these are necessary, but not sufficient, conditions for classical vortices. In particular, although they can be satisfied in the self-consistent linear approximation that will be outlined below, only the full nonlinearity of the system can establish classical profiles. We will term such zeros as satisfy these conditions protovortices. It has to be said that most (but not all [22, 23]) numerical lattice simulations cannot distinguish between protovortices and classical vortices.

5.2. TDLG Condensed Matter

We begin with condensed matter, which we will find to be easier. As the system evolves away from the transition time, the free equation (13) ceases to be valid. In order to retain some analytic understanding of the way that the density is such an ideal quantity for which to make predictions, we adopt the approximation of preserving Gaussian fluctuations by linearizing the self-interaction as

$$\dot{\phi}_a(\mathbf{x}, t) = -[-\nabla^2 + \epsilon_{\text{eff}}(t)]\,\phi_a(\mathbf{x}, t) + \bar{\eta}_a(\mathbf{x}, t) \tag{30}$$

where ϵ_{eff} contains a (self-consistent) term $O(\bar{\beta}\langle|\phi|^2\rangle)$, in which $\bar{\beta}$ is the rescaled coupling. Additive renormalization is necessary, so that $\epsilon_{\text{eff}} \approx \epsilon$, as given earlier, for $t \leq t_0$.

Self-consistent linearization is the standard approximation in nonequilibrium QFT [36, 37], but is not strictly necessary here since numerical simulations that identify line zeros of the field can be made that use the full self-interaction [6]. However, there are none that address our particular problems exactly. Given the similarities with the QFT case, for which it is difficult to do much better than a Gaussian, there are virtues in comparing the Gaussian approximation for the two cases.

The solution for $G(r, t)$ is a straightforward generalization of Eq. (15),

$$G(r, t) = \int_0^\infty d\tau\,\bar{T}(t - \tau/2)(1/(4\pi\tau))^{3/2} \times \exp(-r^2/4\tau)\,\exp\left[-\int_0^\tau ds\,\epsilon_{\text{eff}}(t - s/2)\right] \tag{31}$$

where $\bar{T}$ is the rescaled temperature, as before.

Assuming a *single* zero of $\epsilon_{\text{eff}}(t)$ at $t = t_0$ at $r = 0$ the exponential in the integrand peaks at $\tau = \bar{\tau} = 2(t - t_0)$, the counterpart of the Bragg peak in proper time. Expanding about $\bar{\tau}$ to quadratic order gives [17]

$$G_l(0, t) \approx \bar{T}_c \exp\left[2 \int_{t_0}^{t} du \, |\epsilon_{\text{eff}}(u)|\right] \times \int_0^\infty \frac{d\tau \exp[-(\tau - 2(t - t_0))^2|\epsilon'(t_0)|/4]}{[4\pi(\tau + \bar{l}^2)]^{3/2}} \quad (32)$$

where we have put in the momentum cutoff $k^{-1} > l = \bar{l}\xi_0 = O(\xi_0)$ of Eq. (27) by hand. For times $t > \epsilon_0\tau_Q$ we see that, as the unfreezing occurs, long-wavelength modes with $k^2 < t/\tau_Q - \epsilon_0$ grow exponentially.

For the backreaction to stop the growth of $G_l(0, t) - G_l(0, t_0)$ at its symmetry-broken value $\bar{\beta}^{-1}$ we must have $\lim_{u\to\infty} \epsilon_{\text{eff}}(u) = 0$, thereby preserving Goldstone's theorem.

Even though the field is correlated over a distance $\bar{\xi} \gg l$ at $t = t_0$, the density of line zeros $n_{\text{zero}} = O(l^{-2})$ depends entirely on the scale at which we look. In no way would we wish to identify these line zeros with prototype vortices. However, as time passes the peak of the exponential grows and n_{zero} becomes increasingly insensitive to l. How much time we have depends on the magnitude of $\bar{\beta}$, since once $G(0, t)$ has reached this value it stops growing. The time t^* at which this happens can be estimated by substituting $\epsilon(u)$ for $\epsilon_{\text{eff}}(u)$ in the expression for $G_l(0, t)$ above.

For $t > t^*$ a more careful analysis shows that $G_l(0, t)$ can be written as

$$G_l(0, t) \approx \int_0^\infty \frac{d\tau \, \bar{T}(t - \tau/2)}{[4\pi(\tau + \bar{l}^2)]^{3/2}} \bar{G}(\tau, t) \quad (33)$$

where $\bar{G}(\tau, t)$ has the same peak as before at $\tau = 2(t - t_0)$, in magnitude and position, but $\bar{G}(\tau, t) \cong 1$ for $\tau < 2(t - t^*)$. Thus, for $\tau_Q \gg \tau_0$, $G_l(0, t)$ can be approximately separated as $G_l(0, t) \cong G_l^{\text{UV}}(t) + G^{\text{IR}}(t)$, where

$$G_l^{\text{UV}}(t) = \bar{T}(t) \int_0^\infty d\tau/[4\pi(\tau + \bar{l}^2)]^{3/2} \quad (34)$$

describes the scale-*dependent* short-wavelength thermal noise, and

$$G^{\text{IR}}(t) = \frac{\bar{T}_c}{(8\pi(t - t_0))^{3/2}} \int_{-\infty}^{\infty} d\tau \, \bar{G}(\tau, t) \quad (35)$$

describes the scale-*independent*, temperature-dependent, long-wavelength fluctuations. A similar decomposition $G''_l(0, t) \cong G''^{\text{UV}}_t(t) + G''^{\text{IR}}(t)$ can be performed. In particular, $G''^{\text{IR}}(t)/G^{\text{IR}}(t) = O(t^{-1})$.

First, suppose that, for $t > t^*$, $G^{\text{IR}}(t) \gg G_l^{\text{UV}}(t)$ and $G''^{\text{IR}}(t) \gg G''^{\text{UV}}_l(t)$, as would be the case for a temperature quench $\bar{T}(t) \to 0$. Then, with little thermal noise, we have widely separated line zeros, with density $n_{\text{zero}}(t) \approx$

$-G''^{\rm IR}(t)/2\pi G^{\rm IR}(t)$ and $\partial n_{\rm zero}/\partial l$ is small in comparison to $n_{\rm zero}/l$ at $l = \xi_0$. Further, once the line zeros have straightened on small scales at $t > t^*$, the Gaussian field energy, largely in field gradients, is

$$\overline{F} \approx \left\langle \int_V d^3x \, \frac{1}{2} \, (\nabla\phi_a)^2 \right\rangle = -VG''(0, t) \tag{36}$$

where V is the spatial volume. This matches the energy

$$\overline{E} \approx Vn_{\rm def}(t)(2\pi G(0, t)) = -VG''(0, t) \tag{37}$$

possessed by a network of classical global strings with density $n_{\rm zero}$, in the same approximation of cutting off their logarithmic tails.

From our comments above, we identify such essentially nonfractal line zeros with prototype vortices, and $n_{\rm zero}$ with $n_{\rm def}$. Of course, we require non-Gaussianity to create true classical energy profiles [23].

For times $t > t^*$

$$n_{\rm def}(t) \approx \frac{\bar{t}}{8\pi(t - t_0)} \frac{1}{\xi_0^2} \sqrt{\frac{\tau_0}{\tau_Q}} \tag{38}$$

which is the solution to Vinen's equation [24]

$$\frac{\partial n_{\rm def}}{\partial t} = -\chi_2 \, \frac{\hbar}{m} \, n_{\rm def}^2 \tag{39}$$

where $\chi_2 = 4\pi$ in our approximation.[6] What is remarkable in this approximation is that the density of line zeros uses *no* property of the self-mass contribution to $\epsilon_{\rm eff}(t)$, self-consistent or otherwise.

For ^{3}He, with negligible UV contributions, we estimate the primordial density of vortices as

$$n_{\rm def}(t^*) \approx \frac{\bar{t}}{8\pi(t^* - t_0)} \frac{1}{\xi_0^2} \sqrt{\frac{\tau_0}{\tau_Q}} \tag{40}$$

in accord with the original prediction of Zurek. Because of the rapid growth of $G(0, t)$, $(t^* - t_0)/\bar{t} = p > 1 = O(1)$. We note that the factor[7] of $f^2 = 8\pi p$ gives a value of $f = O(10)$, in agreement with the empirical results of ref. 13 and the numerical results of ref. 25.[8]

Whereas Eq. 5, (40) is appropriate for ^{3}He, the situation for the Lancaster ^{4}He experiments is complex, since they are *pressure* quenches for which the

[6] Calculations for χ_2 for realistic values of ξ_0 and τ_0 give $\chi_2 > 4\pi$ for both ^{4}He and ^{3}He. This is much larger than the empirical value [15] $\chi_2 \approx 0.005$ from turbulent flow experiments.

[7] An erroneous factor of 3 appeared in the result of ref. 16.

[8] The temperature quench of the latter is somewhat different from that considered here, but should still give the same results in this case.

temperature T is almost constant at $T \approx T_c$. Unlike temperature quenches [6, 26], thermal fluctuations here remain at full strength.[9] The necessary time *independence* of $G^{IR}(t)$ for $t > t^*$ is achieved by taking $\epsilon_{eff}(u) = O(u^{-1})$. In consequence, as t increases beyond t^* the relative magnitude of the UV and IR contributions to $G_l(0, t)$ remains *approximately constant* at its value at $t = t^*$.

Nonetheless, as long as the UV fluctuations are insignificant at $t = t^*$ the density of line zeros will remain largely independent of scale. This follows if $G''^{IR}(t^*) \gg G''^{UV}_l(t^*)$, since $G''_l(0, t)$ becomes scale independent later than $G_l(0, t)$. In ref. 16 we showed that this is true provided

$$(\tau_Q/\tau_0)(1 - T_G/T_c) \leq C\pi^4 \tag{41}$$

where $C = O(1)$ and T_G is the Ginzburg temperature. With $\tau_Q/\tau_0 = O(10^3)$ and $(1 - T_G/T_c) = O(10^{-12})$ this inequality is well satisfied for a linearized TDLG theory for ^{3}He derived[10] from the full TDGL theory [10], but there is no way that it can be satisfied for ^{4}He when subjected to a slow mechanical quench, as in the Lancaster experiment, for which $\tau_Q/\tau_0 = O(10^{10})$ since the Ginzburg regime is so large that $(1 - T_G/T_c) = O(1)$. As far as the left-hand side of Eq. (41) is concerned, the ^{4}He quench is *19* orders of magnitude slower than its ^{3}He counterpart.

When the inequality is badly violated, as with ^{4}He, the density of zeros $n_{zero} = O(l^{-2})$ after t^* again depends explicitly on the scale l at which we look and they are not candidates for vortices. Since the whole of the quench takes place within the Ginzburg regime this is not implausible. Even if we suppose that n_{def} above is a starting point for calculating the density at later times, albeit with a different t_0, thereby preserving Vinen's law, we then have the earlier problem of the large $\chi_2 = O(f^2)$, which would make it almost impossible to see vortices.

For all that, a numerical simulation that goes beyond the Gaussian approximation specifically tailored to the Lancaster parameters is crucial if we are to understand what is really happening. We hope to pursue this elsewhere.

6. THE APPEARANCE OF STRUCTURE IN QFT

When in Section 5.2 we set up the closed-time-path formalism for the field probabilities $p_t[\Phi]$, our aim was the limited one of establishing the role of Kibble's causal correlation length $\bar{\xi}$ in Eq. (26) We now appreciate, from

[9] Even for ^{3}He, T/T_c never gets very small, and henceforth we take $T = T_c$ in $G_i(0,t)$ above.
[10] Ignoring the position-dependent temperature of ref. 12.

condensed matter theory, that this does not, of itself, imply vortices at that separation.

6.1. Protovortices in QFT

To establish a link between the correlation function $G(r, t)$ and vortices is even more problematic in QFT than for condensed matter systems. Yet again, we attempt to count vortices by counting line zeros [27]. In the Gaussian approximations that we shall continue to adopt, the expression (28) for $n_{\rm zero}$ is equally applicable to QFT. This counting of zeros is the basis of numerous numerical simulations [28–30] of cosmic string networks built from Gaussian fluctuations.

The prerequisites for line zeros in condensed matter that we posed after Eq. (28) still stand for QFT (except that $\langle|\phi^2|\rangle = M^2/\lambda$), but there are further complications peculiar to QFT. In particular, in QFT we need to consider the whole density matrix $\langle\Phi'|\rho(t)|\Phi\rangle$ rather than just the diagonal elements $p_t[\Phi] = \langle\Phi|\rho(t)|\Phi\rangle$. Classicality is understood in terms of "decoherence" manifest most simply by the approximate diagonalization of the reduced density matrix on coarse-graining. By this we mean the separation of the whole into the "system" and its "environment" whose degrees of freedom are integrated over, to give a reduced density matrix. The environment can be either other fields with which our scalar is interacting or even the short-wavelength modes of the scalar field itself [31, 32]. When interactions are taken into account this leads to quantum noise and dissipation.

In the Gaussian approximations that we shall adopt here, with $\langle\Phi\rangle = 0$, integrating out short wavelengths with $k > l^{-1}$ is just equivalent to a momentum cutoff at the same value. This gives neither noise nor dissipation and diagonalization does not occur. Nonetheless, from our viewpoint of counting line zeros, fluctuations are still present when $l = O(M^{-1})$ that can prevent us from identifying line zeros with protovortices if the quenches are too slow.

For all these caveats, there are other symptoms of classical behavior once $G_l(0; t)$ is nonperturbatively large. Instead of a field basis, we can work in a particle basis and measure the particle production as the transition proceeds. The presence of a nonperturbatively large peak in $k^2G(k; t)$ at $k = k_0$ signals a nonperturbatively large occupation number $N_{k_0} \propto 1/\lambda$ of particles at the same wavenumber k_0 [36]. With $n_{\rm zero}$ of (28) of order k_0^2 this shows that the long-wavelength modes can now begin to be treated classically. From a slightly different viewpoint, the Wigner functional only peaks about the classical phase-space trajectory once the power is nonperturbatively large [33, 34]. More crudely, the diagonal density matrix elements are only then significantly nonzero for nonperturbatively large field configurations $\phi \propto \lambda^{-1/2}$, like vortices.

6.2. Mode Growth Versus Fluctuations

For early times we revert to the mode decomposition of Eq. (24). The term coth $(\omega_{\text{in}}/2T_0)$ appearing in it can be approximated by $2T_0/\sqrt{\epsilon_0}M$. Even though this is a temperature quench, it shows strong similarities to the pressure quench of condensed matter since both the long- and short-wavelength contributions to $G(r, t)$ are scaled by the same temperature and we cannot switch off the latter.

The field becomes ordered, as before, because of the exponential growth of long-wavelength modes, which stop growing once the field has sampled the ground states. What matters is the relative weight of these modes (the "Bragg" peak) to the fluctuating short-wavelength modes in the decomposition (25) at this time since the contribution of these latter is very sensitive to the cutoff l. Only if their contribution to Eq. (8) is small when field growth stops can a network of line zeros be well defined at early times, let alone have the predicted density. Since the peak is nonperturbatively large this requires small coupling, which we assume.

Consider a quench with $\epsilon(t)$ as in Eq. (10), in which the symmetry breaking begins at relative time $\Delta t = t - t_0 = 0$. For a *free* roll, the exponentially growing modes that appear when $\Delta t > t_k^- = t_Q\, k^2/M^2$ lead to the approximate WKB solution [35]

$$G(r; \Delta t) \propto \frac{T}{M|m(\Delta t)}\left(\frac{M}{\sqrt{\Delta t t_Q}}\right)^{3/2} \exp\left(\frac{4M\Delta t^{3/2}}{3\sqrt{t_Q}}\right) \exp\left(-\frac{r^2}{\xi^2(\Delta t)}\right) \qquad (42)$$

where $\xi^2(\Delta t) = 2\sqrt{\Delta t t_Q}/M$. The provisional freeze-in time t_* when $\langle|\phi^2|\rangle = M^2/\lambda$ is then, for $Mt_Q < 1/\lambda$,

$$M\,\Delta t_* \simeq (Mt_Q)^{1/3}\,[\ln(1/\lambda)]^{2/3} \simeq M\bar{t}\,[\ln(1/\lambda)]^{2/3} \qquad (43)$$

where $\Delta t_* = t_* - t_0$. This is greater than $M\bar{t}$, but not by a large multiple. Comparison with condensed matter, for which the ratio is a few (3–5), suggests that we do not need a superweak theory [35].

At this qualitative level the correlation length at t_* is given by

$$M^2\xi^2(t_*) \simeq (Mt_Q)^{2/3}\,[\ln(1/\lambda)]^{1/3} \qquad (44)$$

giving, at $t = t_*$,

$$n_{\text{zero}} = \frac{M^2}{\pi(M\tau_Q)^{2/3}}\,[\ln(1/\lambda)]^{-1/3}[1 + E] \qquad (45)$$

The error term $E = O\,(\lambda^{1/2}(Mt_Q)^{4/3}\,[\ln(1/\lambda)]^{-1/3})$ is due to the ever-present thermal fluctuations, sensitive to the cutoff. In mimicry of Eq. (8), it is helpful to rewrite Eq. (45) as

$$n_{\rm zero} = \left[\frac{1}{\pi\xi_0^2}\left(\frac{\tau_0}{\tau_Q}\right)^{2/3}\right][\ln(1/\lambda)]^{-1/3}[1 + E] \tag{46}$$

in terms of the scales $\tau_0 = \xi_0 = M^{-1}$. The first term in Eq. (46) is the Kibble estimate of Eq. (8), and the second is a small multiplying factor, which yet again shows that the estimate can be correct, but for completely different reasons. The third term shows when it can be correct, since E is also a measure of the sensitivity of $n_{\rm def}$ to the scale at which it is measured. The condition $E^2 \ll 1$, necessary for a protovortex network to be defined, is then guaranteed if

$$(\tau_Q/\tau_0)^2(1 - T_G/T_c) < C \tag{47}$$

$C = O(1)$, on using the relation $1 - T_G/T_c = O(\lambda)$. This is the QFT counterpart to Eq. (41).

6.3. Backreaction in QFT

To improve upon the free-roll result more honestly, but retain the Gaussian approximation for the field correlation functions, the best we can do is adopt a mean-field approximation along the lines of refs. [36 and 37], as we did for the CM systems earlier. As there, it does have the correct behavior of stopping domain growth as the field spreads to the potential minima. As before, only the large-N expansion preserves Goldstone's theorem.

$G(r;t)$ still has the mode decomposition of Eq. (25), but the modes $\chi_k^{\pm}$ now satisfy the equation

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 + m^2(t) + \lambda\langle\Phi^2(\mathbf{0})\rangle_t\right]\chi_k^{\pm}(t) = 0 \tag{48}$$

where we have taken $N = 2$. Because $\lambda\phi^4$ theory is not asymptotically free, particularly in the Hartree approximation, the renormalized λ coupling shows a Landau ghost. This means that the theory can only be taken as a low-energy effective theory.

The end result, on making a single subtraction at $t = 0$, is [35]

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 + m^2(t) + \lambda\int d\!\!{}^-{}^3p\; C(p)[\chi_p^+(t)\chi_p^-(t) - 1]\right]\chi_k^{\pm}(t) = 0 \tag{49}$$

which we write as

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 - \mu^2(t)\right]\chi_k(t) = 0 \tag{50}$$

On keeping just the unstable modes in $\langle\Phi^2(\mathbf{0})\rangle_t$, then, as it grows, its contribution to (49) weakens the instabilities, so that only longer wavelengths become unstable. At t^* the instabilities shut off, by definition, and oscillatory behavior ensues. Since the mode with wavenumber $k > 0$ stops growing at time $t_k^+ < t^*$, where $\mu^2(t_k^+) = \mathbf{k}^2$, the free-roll density at t^* must be an overestimate.

An approximation that improves upon the WKB approximation is

$$\chi_k(t) \approx \left(\frac{\pi M}{2\Omega_k(\eta)}\right)^{1/2} \exp\left(\int_0^t dt\ \Omega(t)\right) \tag{51}$$

when $\eta = M(t_k^+ - t) > 0$ is large, and $\Omega_k^2(t) = \mu^2(t) - \mathbf{k}^2$. On expanding the exponent in powers of k and retaining only the quadratic terms, we recover the WKB approximation when $\mu(t)$ is nonzero. The result is to show that the backreaction has little effect for times $t < t^*$. For $t > t^*$ oscillatory modes take over the correlation function and we expect oscillations in $G(k;\, t)$.

In practice the backreaction rapidly forces $\mu^2(t)$ toward zero if the coupling is not too small [36]. This requires that we graft purely oscillatory long-wavelength behavior onto the nonperturbatively large exponential mode

$$\chi_k^+(t^*) \approx \alpha_k \exp\left(\int_0^{t^*} dt'\ \mu(t')\right)\exp\left(-\frac{\sqrt{\tau_Q t^*}}{M}\, k^2\right) \tag{52}$$

The end result is a new power spectrum, obtained by superimposing oscillatory behavior onto the old spectrum, frozen at time t^*. As a gross oversimplification, the contribution from the earlier exponential modes alone can only be to contribute terms something like

$$G(r;\, t) \propto \frac{T}{M|m(t^*)|}\, e^{4M(t^*)^{3/2}/3\sqrt{\tau_Q}} \int_{|\mathbf{k}|<M} d\!\!\!{}^-{}^3k\ e^{i\mathbf{k.x}} e^{-2\sqrt{t^*\tau_Q}k^2/M} \times \left[\cos k(t - t^*) + \frac{\Omega(k) - W'(k)}{k} \sin k(t - t^*)\right]^2 \tag{53}$$

to G, where $\Omega = M(t^* - t_k)^{1/2}/\tau_Q^{1/2}$ and $W' = 1/4(t^* - t_k)$. The details are almost irrelevant, since the density of line zeros is independent of the normalization, and only weakly dependent on the power spectrum.

The $k = 0$ mode of Eq. (53) encodes the simple solution $\chi_{k=0}(t) = a + bt$ when $\mu^2 = 0$. As observed by Boyanovsky *et al.* [26], this has built into it the basic causality discussed by Kibble [3]. Specifically, for $r,t \to \infty$, but $r/2t$ constant ($\neq 1$),

$$G(r,\, t) \approx (C/r)\Theta(2t/r - 1) \tag{54}$$

It follows directly that this causality, engendered by the Goldstone particles of the self-consistent theory, has little effect on the density of line zeros that we expect to mature into fully classical vortices, since that is determined by the behavior at $r = 0$.

Further, for large t the power spectrum effectively has a k^{-2} behavior for small k, unlike the white noise that would follow from Eq. (42) It has been suggested [29] that for such a spectrum most, if not all, of the vortices are in loops, with little or no self-avoiding "infinite" string (but see ref. 30). If there was no infinite string the evolution of the network could be very different [38] from that of white noise, where approximately 75% of the string is "infinite" [28]. Although causality due to massless Goldstone modes is unrealistic, the linking of causal horizons to the long-wavelength spectrum is general. It has to be said that this approximation should not be taken very seriously for large t on different grounds, since we would expect rescattering to take place at times $\Delta t = O(1/\lambda)$ in a way that is precluded by the Gaussian approximation.

Returning to our original concerns, if Eq. (47) is not satisfied, it is difficult to imagine how clean vortices, or protovortices, can appear later without some additional ingredient.

7. CONCLUSIONS

We examined the Kibble/Zurek predictions for the onset of phase transitions and the appearance of defects (in particular, vortices or global cosmic strings) as a signal of the symmetry breaking. Our results are in agreement with their predictions, Eqs. (4) and (7), as to the magnitude of the correlation length at the time the transition truly begins, equally true for condensed matter and QFT.

However, this is not simply a measure of the separation of defects at the time of their appearance. The time $\bar{t}$ is too early for the field to have found the true ground states of the theory. We believe that time, essentially the spinodal time, is the time at which protovortices can appear, which can later evolve into the standard classical vortices of the theory.

Even then, they may not appear because of thermal field fluctuations. In TDLG condensed matter thermal noise is proportional to temperature. If temperature is *fixed*, but *not* otherwise, as in the pressure quenches of ^{4}He, this noise can inhibit the production of vortices, although there are other factors to be taken into account (such as their decay rate). On the other hand, on quenching from a high temperature in QFT there are always thermal fluctuations, and these can also disturb the appearance of vortices. The condition that thermal fluctuations are ignorable at the time that the field has achieved the true ground states can be written

$$(\tau_Q/\tau_0)^\gamma(1 - T_G/T_c) < C \qquad (55)$$

where $\gamma = 1$ for condensed matter and $\gamma = 2$ for QFT. Here $C = O(1)$.

Equivalently, this can be understood as the condition that the density of line zeros is insensitive to the scale $O(\xi_0)$ at which they are viewed, as would happen for a classical vortex network.

This restores the role of the Ginzburg temperature T_G that the simple causal arguments overlooked, but does not restore thermal fluctuations as the exclusive agent *for* vortex production, as happened in early arguments. Quenches in ^{4}He provide the major example for which Eq. (55) is not satisfied.

What happens at late time is unclear, although for the TDLG, numerical simulations can be performed (but have yet to address this problem exactly). On the other hand, not only is the case of a single self-interacting *quantum* scalar field in flat space-time a caricature of the early universe, but it is extremely difficult to go beyond the Gaussian approximation. To do better requires that we do differently. There are several possible approaches. One step is to take the FRW metric of the early universe seriously, whereby the dissipation due to the expansion of the universe can change the situation dramatically [39]. Other approaches are more explicit in their attempts to trigger decoherence explicitly, as we mentioned earlier. Most simply, the short-wavelength parts of the field can be treated as an environment to be integrated over, to give a coarse-grained theory of long-wavelength modes acting classically in the presence of noise. However, such noise is more complicated than in TDLG theory, being multiplicative as well as additive, and colored [40, 32, 31]. This is an area to be pursued elsewhere.

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REFERENCES

1. T. W. B. Kibble, *J. Phys. A* **9**, 1387 (1976).
2. T. W. B. Kibble, In Common trends in particle and condensed matter physics, *Phys. Re.* **67**, 183 (1980).
3. T. W. B. Kibble, Talk presented at the Meeting on Monopoles in Quantum Field Theory at ICTP, Trieste, December 1991; Imperial College preprint.
4. W. H. Zurek, *Nature* **317**, 505 (1985); *Acta Phys. Polon. B* **24**, 1301 (1993); see also W. H. Zurek, *Phys. Rep.* **276** (4), 177 (1996).
5. R. J. Rivers, E. Kavoussanaki, and G. Karra, *Condensed Matter Phys.*, to appear.

6. W. H. Zurek, *Nature* **382**, 297 (1996); P. Laguna and W. H. Zurek, *Phys. Rev. Lett.* **78**, 2519 (1997); A. Yates and W. H. Zurek, *Phys. Rev. Lett.* **80**, 5477 (1998).
7. J. Garcia-Bellido and A. Linde, *Phys. Rev. D* **57**, 6075 (1998).
8. E. J. Copeland, D. Haws, S. Holbraad, and R. J. Rivers, *Physica A* **179**, 507 (1991).
9. M. M. Salomaa and G. E. Volovik, *Rev. Mod. Phys.* **59**, 533 (1987).
10. Yu. M. Bunkov and O. D. Timofeevskaya, *Phys. Rev. Lett.* **80**, 4927 (1998).
11. V. M. H. Ruutu *et al.*, *Nature* **382**, 334 (1996).
12. I. S. Aranson, N. B. Kopnin, and V. M. Vinokur, preprint cond-mat/9905286.
13. C. Bauerle *et al.*, *Nature* **382**, 332 (1996).
14. P. C. Hendry *et al.*, *Nature* **368**, 315 (1994).
15. M. E. Dodd *et al.*, *Phys. Rev. Lett.* **81**, 3703 (1998); *J. Low Temp. Phys.* **15**, 89 (1999).
16. G. Karra and R. J. Rivers, *Phys. Rev. Lett.* **81**, 3707 (1998).
17. R. J. Rivers, *Phys. Rev. Lett.*, **84**, 1248 (2000).
18. M. Bowick and A. Momen, *Phys. Rev. D* **58**, 085014 (1998).
19. V. N. Popov, *Functional Integrals and Collective excitations* (Cambridge University Press, Cambridge, 1997), Chapter 8.
20. B. I. Halperin, In *Physics of Defects*, R. Balian, M. Kléman, and J. -P. Poirier, eds. (North-Holland, Amsterdam, 1981), p. 816.
21. F. Liu and G. F. Mazenko, *Phys. Rev. B* **46**, 5963 (1992).
22. M. Gleiser and H. -R. Muller, *Phys. Lett. B* **422**, 69 (1998).
23. D. Ibaceta and E. Calzetta, *Phys. Rev. E* **60**, 2999 (1999).
24. W. F. Vinen, *Proc. R. Soc. Lond. A* **242**, 493 (1957).
25. N. D. Antunes, L. M. A. Bettencourt, and W. H. Zurek, *Phys. Rev. Left.* **82**, 2824 (1999), and references therein.
26. D. Boyanovsky, H. J. de Vega, and R. Holman, preprint hep-ph/9903534; in *Proceedings of the NATO Advanced Study Institute: Topological Defects and the Non-Equilibrium Dynamics of Symmetry Breaking Phase Transitions*, to appear.
27. A. J. Gill and R. J. Rivers, *Phys. Rev. D* **51**, 6949 (1995).
28. T. Vachasparti and A. Vilenkin, *Phys. Rev. D* **30**, 2036 (1984).
29. J. Robinson and A. Yates, *Phys. Rev. D* **54**, 5211 (1996).
30. R. J. Scherrer and A. Vilenkin, *Phys. Rev. D* **56**, 647 (1997); **58**, 103501 (1998); N. D. Antunes, M. A. Bettencourt, and A. Yates, e-print hep-ph/9901391; submitted for publication.
31. C. Greiner and B. Muller, *Phys. Rev. D* **55**, 1026 (1997).
32. F. Lombardo and F. D. Mazzitelli, *Phys. Rev. D* **53**, 2001 (1996).
33. A. Guth and S.-Y. Pi, *Phys. Rev. D* **32**, 1899 (1985); O. Eboli, R. Jackiw, and S.-Y. Pi, *Phys. Rev. D* **37**, 3557 (1988).
34. S. Mrowczynski and B. Muller, *Phys. Rev. D* **50**, 7542 (1994).
35. G. Karra and R. J. Rivers, *Phys. Lett. B* **414**, 28 (1997).
36. D. Boyanovsky, Da-Shin Lee, and A. Singh, *Phys. Rev. D* **48**, 800 (1993); D. Boyanovsky, H. J. de Vega, and R. Holman, *Phys. Rev. D* **49**, 2769 (1994).
37. F. Cooper, S. Habib, Y. Kluger, and E. Mottola, *Phys. Rev. D* **55**, 6471 (1997); F. Cooper, S. Habib, Y. Kluger, E. Mottola, J. P. Paz, and P. R. Anderson, *Phys. Rev. D* **50**, 2848 (1994).
38. E. J. Copeland, T. W. B. Kibble, and D. A. Steer, *Phys. Rev. D* **58**, 043508 (1998).
39. G. J. Stephens, E. A. Calzetta, B.-L. Hu, and S. A. Ramsey, *Phys. Rev. D* **59**, 045009 (1999).
40. M. Gleiser and R. O. Ramos, *Phys. Rev. D* **50**, 2441 (1994); A. Berera, M. Gleiser, and R. O. Ramos, *Phys. Rev. D* **58**, 123508 (1998).